

QUASILINEAR ELLIPTIC EQUATIONS OF BORN-INFELD TYPE

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ABSTRACT. This paper is devoted to the study, with variational technique, of the following quasilinear elliptic problem:

$$\begin{cases} -\Delta_p u - \beta \Delta_q u = g(u) & \text{in } \mathbb{R}^N, \\ u(x) \rightarrow 0 & \text{as } |x| \rightarrow +\infty, \end{cases}$$

where $N \geq 3$, $1 < p < q$ and $p < N$. We are interested in the existence of positive solutions for general nonlinearities. Especially we obtain the existence result for the zero mass case, which includes a large class of pure power nonlinearities. More general quasilinear problems of Born-Infeld type are also considered.

1. INTRODUCTION

In this paper, we study, with variational technique, the following quasilinear elliptic problem:

$$(\mathcal{P}) \quad \begin{cases} -\Delta_p u - \beta \Delta_q u = g(u) & \text{in } \mathbb{R}^N, \\ u(x) \rightarrow 0 & \text{as } |x| \rightarrow +\infty, \end{cases}$$

where $N \geq 3$, $\beta > 0$, $1 < p < q$, $p < N$ and $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the p -Laplacian. In the last decades, a lot of works has been done for the study of (p, q) -Laplace equation. However, most of them are devoted to the bounded domain case or problems with critical nonlinearities (see, for example, [9, 15, 19, 26] and references therein).

If $\beta = 0$, (\mathcal{P}) reduces to the following scalar field equation:

$$(1.1) \quad -\Delta_p u = g(u) \text{ in } \mathbb{R}^N.$$

The existence of solutions of (1.1) has been studied, among others, in [11, 17, 18]. Moreover when $p = 2$, the almost optimal condition for the existence of nontrivial solutions has been obtained in [11]. However, a scaling property which plays an essential role in [11, 17] is lost if $\beta \neq 0$ in (\mathcal{P}) , causing that the approach in [11, 17] cannot be applied to (\mathcal{P}) . Thus it is an challenging problem to look for an optimal condition for the existence of nontrivial solutions of (\mathcal{P}) .

The aim of this paper is, therefore, to consider (\mathcal{P}) in the whole \mathbb{R}^N and for a general nonlinearity g . Especially, we do not assume any monotonicity conditions on g .

Our another motivation comes from the study of the *Born-Infeld* equation which appears in electromagnetism:

$$(1.2) \quad -\operatorname{div} \left(\frac{\nabla u}{\sqrt{1 - \frac{1}{b^2} |\nabla u|^2}} \right) = g(u) \text{ in } \mathbb{R}^N,$$

where b is a positive constant and called *Born's aether constant*. (We refer to [12] and references therein for more physical backgrounds of the Born-Infeld equation.) Indeed by the Taylor expansion, it follows that

$$\frac{1}{\sqrt{1-x}} = 1 + \frac{x}{2} + \frac{3}{2 \cdot 2^2} x^2 + \frac{5!!}{3! \cdot 2^3} x^3 + \cdots + \frac{(2k-3)!!}{(k-1)! 2^{k-1}} x^{k-1} + \cdots \quad \text{for } |x| < 1.$$

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Putting $x = \frac{|\nabla u|^2}{b^2}$ and $\beta = \frac{1}{2b^2}$ formally, we can see that the 0-th order approximated problem of (1.2) is exactly the scalar field equation (1.1) with $p = 2$. When we adopt the 1-st order approximation, one has the following quasilinear elliptic equation:

$$-\Delta u - \beta \Delta_4 u = g(u) \quad \text{in } \mathbb{R}^N,$$

which can be obtained by taking $p = 2$ and $q = 4$ in (P). Furthermore the k -th order approximated problem is given by

$$(1.3) \quad -\Delta u - \beta \Delta_4 u - \frac{3}{2}\beta^2 \Delta_6 u - \dots - \frac{(2k-3)!!}{(k-1)!} \beta^{k-1} \Delta_{2k} u = g(u) \quad \text{in } \mathbb{R}^N,$$

where $k \in \mathbb{N}$, $(2k-3)!! = (2k-3)(2k-5)\dots 5 \cdot 3 \cdot 1$, $(-1)!! = 1$. Thus it is natural to ask if solutions of (1.2) can be obtained as a limit of solutions for (1.3). This question has been considered in [12] for the inhomogeneous Born-Infeld problem:

$$(1.4) \quad -\operatorname{div} \left(\frac{\nabla u}{\sqrt{1 - \frac{1}{b^2} |\nabla u|^2}} \right) = \rho(x) \quad \text{in } \mathbb{R}^N.$$

It is shown that under suitable assumptions on ρ , the unique minimizer of the action functional associated to (1.4) can be obtained as a weak limit of the unique solution of the k -th order approximated problem for (1.4). (See [12, Theorem 5.2].) On the other hand, problem (1.2) is much less studied. In [13], the case $g(u) = |u|^{\alpha-2}u$ with $\alpha > \frac{2N}{N-2}$ has been considered. Then it was shown that (1.2) has a positive radial solution and a sequence of radial solutions. Moreover, again by restricting the research to solutions with radial symmetry, in [2] the equation (1.2) is reduced to an ODE for which the existence, non-existence and multiplicity of ground states (namely positive solutions going to zero at infinity) and bound states (i.e. solutions going to zero at infinity) are investigated for the Lane-Emden type equation. By the use of the shooting method, in [1] the existence of a ground state solution is also determined for the equation presenting a sign-changing nonlinearity.

Our purpose of this paper is to investigate the existence of positive solutions of (P) and (1.3) for general nonlinearities including the case $g(u) = u^\alpha$. We expect our existence results will be the next step for the further study of the Born-Infeld equation (1.2). Hereafter in this paper, we take $\beta = 1$ for simplicity, since β plays no essential role in the study of the existence of solutions.

In order to consider general nonlinear terms, we have to take into account behavior of $g(s)$ near zero and infinity. For the problem (1.1) with $p = 2$ and in the *positive mass* case, namely when $g(s)$ satisfies

$$-\infty < \liminf_{s \rightarrow 0^+} \frac{g(s)}{s} \leq \limsup_{s \rightarrow 0^+} \frac{g(s)}{s} = -m \text{ for some } m > 0,$$

almost optimal condition for the existence of nontrivial solutions has been obtained in [11]. (See also [23].) Conversely, whenever $m = 0$, the so called *zero mass* case, some results are contained, among others, in [32], if g corresponds to the critical power $s^{(N+2)/(N-2)}$, and in [4, 8, 11], if g is supercritical near the origin and subcritical at infinity (see also [10] for the case of exterior domain and [5] for complex valued solutions).

We anticipate that our problem has two quite interesting features: we can treat the zero mass case and the positive mass one in a similar way and, moreover, in the zero mass case, we can treat several pure power nonlinearities. This is due to the particular functional setting that we will introduce to study (P). Indeed, while in presence of a single p -laplacian the natural framework is $D^{1,p}(\mathbb{R}^N)$, namely the completion of $C_0^\infty(\mathbb{R}^N)$ with the respect of the L^p -norm of the gradient, and we know that $D^{1,p}(\mathbb{R}^N)$ is embedded only into $L^{p^*}(\mathbb{R}^N)$, where $p^* = (pN)/(N-p)$, in our case we will introduce a combination of Sobolev spaces, a sort of intersection between $D^{1,p}(\mathbb{R}^N)$ and $D^{1,q}(\mathbb{R}^N)$, which guarantees suitable embeddings properties into a large range of Lebesgue spaces (see Section 2.1 for more details). Finally, we would like

to stress that the unique assumption on q is that it is strictly greater than p but it can be large as we want, as this is a great help in order to better approximate the Born-Infeld operator.

We can now introduce our precise assumptions and results and we start dealing with the zero mass case. The following hypotheses can be regarded as a natural extension of the zero mass case for (1.1) to the quasilinear problem (\mathcal{P}).

On the nonlinearity g , we require that

(g1) $g \in C(\mathbb{R}, \mathbb{R})$, $g(s) \equiv 0$ if $s \leq 0$;

(g2) for all $\ell \in [p, p^*]$, it holds

$$-\infty \leq \limsup_{s \rightarrow 0^+} \frac{g(s)}{s^{\ell-1}} \leq 0,$$

where $p^* = \frac{pN}{N-p} \in (p, +\infty)$;

(g3) if $q < N$, it holds that

$$(1.5) \quad -\infty \leq \limsup_{s \rightarrow +\infty} \frac{g(s)}{s^{q^*-1}} \leq 0,$$

where $q^* = \frac{qN}{N-q} \in (p^*, +\infty)$; instead, if $q \geq N$, we assume (1.5) holds for some $q^* > \max\{q, p^*\}$;

(g4) there exists $\zeta > 0$ such that $G(\zeta) = \int_0^\zeta g(s) ds > 0$.

As we will see in Section 2.1, the exponents p^* and q^* appear naturally if we consider embedding theorems for energy spaces associated with (\mathcal{P}). Especially, if $q < N$, since q^* can be seen as a critical exponent for (\mathcal{P}), the condition (g3) implies that the nonlinear term $g(s)$ has $W^{1,q}$ -subcritical growth at infinity. On the other hand, (g2) means that $g(s)$ has zero mass, as well as $W^{1,p}$ -supercritical growth near zero. Here we list typical examples of $g(s)$.

- $g(s) = \min\{|s|^{q^*-2}s, |s|^{\ell-2}s\}$ for $p^* < \ell < q^*$.
- $g(s) = |s|^{\ell-2}s$ for $p^* < \ell < q^*$.
- $g(s) = K|s|^{\ell_1-2}s - |s|^{\ell_2-2}s$ for $p^* < \ell_1 \leq q^*$, $\ell_1 < \ell_2$ and large $K > 0$.
- $g(s) = -|s|^{\ell_1-2}s + |s|^{\ell_2-2}s$ for $p^* \leq \ell_1 < \ell_2 < q^*$.

As the second example shows, we can consider a large class of pure power nonlinearities for our problem (\mathcal{P}), which is impossible for (1.2). Especially in the case $q > N$, let $\ell > \frac{pN}{N-p}$ be arbitrarily given and consider the nonlinear term $g(s) = s^{\ell-1}$. Then choosing any $q^* > \max\{\ell, q\}$, we see that (g1)-(g4) are all satisfied. In this setting, we have the following result.

Theorem 1.1. *Assume (g1)-(g4). Then problem (\mathcal{P}) has a solution which is positive and radially symmetric and belongs to $C_{\text{loc}}^{1,\sigma}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, for some $\sigma \in (0, 1)$.*

Moreover we will prove that there exists a *radial ground state solution*, namely a solution of (\mathcal{P}) which minimizes the action functional among all nontrivial radial solutions of (\mathcal{P}) (see Theorem 2.9 for the precise statement).

Next we state a result for the positive mass case for (\mathcal{P}). In this case, we assume the following condition instead of assumption (g2):

(g2') there exist $\ell \in [p, p^*]$ and $m_\ell > 0$ such that

$$-\infty < \liminf_{s \rightarrow 0^+} \frac{g(s)}{s^{\ell-1}} \leq \limsup_{s \rightarrow 0^+} \frac{g(s)}{s^{\ell-1}} = -m_\ell.$$

Then we obtain the following result.

Theorem 1.2. *Assume (g1), (g2'), (g3) and (g4). Then problem (\mathcal{P}) has a solution which is positive and radially symmetric and belongs to $C_{\text{loc}}^{1,\sigma}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, for some $\sigma \in (0, 1)$.*

Also in this case, the existence of a radial ground state solution of (\mathcal{P}) can be also obtained, see Theorem 3.5 below.

We believe that assumptions (g1), (g2)-(g2'), (g3) and (g4) are almost optimal for (\mathcal{P}) when $q < N$. We finally remark that very general quasilinear elliptic equations have been treated also, for example, in [3, 21, 28], but our problem does not fall in the studied cases.

This paper is organized as follows. In Section 2, we consider the zero mass case. First we prepare embedding theorems for energy spaces associated with (\mathcal{P}) and perform the variational setting in Section 2.1. Next in Section 2.2, we prove the existence of a positive radial solution of (\mathcal{P}) by using the Mountain Pass Theorem together with the Monotonicity trick. We consider the positive mass case in Section 3 and, finally, we devote the Section 4 to the study of the k -th order approximated problem (1.3).

2. THE ZERO MASS CASE

2.1. Variational setting and preliminaries.

In this section, we give some preliminaries. First, we introduce the framework where we will study (\mathcal{P}) and present its embedding properties. Next, we introduce the energy functional associated with (\mathcal{P}) and modify the nonlinear term in order to find a nontrivial critical point.

We work on the functional space $\mathcal{H}_0^{p,q}$ which is given by $\mathcal{H}_0^{p,q} = \overline{C_0^\infty(\mathbb{R}^N)}^{\|\cdot\|_{\mathcal{H}_0^{p,q}}}$, where

$$\|u\|_{\mathcal{H}_0^{p,q}} := \|\nabla u\|_p + \|\nabla u\|_q.$$

In the following theorem we study the embeddings properties of $\mathcal{H}_0^{p,q}$.

Theorem 2.1. *Let $1 < p < q$ and $p < N$. Then*

$$\mathcal{H}_0^{p,q} \hookrightarrow L^r(\mathbb{R}^N), \quad \text{for any } \frac{pN}{N-p} \leq r \begin{cases} \leq \frac{qN}{N-q} & \text{if } q < N, \\ < +\infty & \text{if } q = N, \\ \leq +\infty & \text{if } q > N. \end{cases}$$

Proof. We distinguish three different cases.

CASE $1 < p < q < N$.

By standard Sobolev inequalities, we have that

$$\|u\|_{\frac{pN}{N-p}} \leq C\|\nabla u\|_p \leq C\|u\|_{\mathcal{H}_0^{p,q}}, \quad \|u\|_{\frac{qN}{N-q}} \leq C\|\nabla u\|_q \leq C\|u\|_{\mathcal{H}_0^{p,q}}$$

and so

$$\mathcal{H}_0^{p,q} \hookrightarrow L^{\frac{pN}{N-p}}(\mathbb{R}^N) \cap L^{\frac{qN}{N-q}}(\mathbb{R}^N).$$

CASE $1 < p < q = N$.

Going back the proof of the Sobolev inequality, if $u \in C_0^\infty(\mathbb{R}^N)$, one has

$$(2.1) \quad \|u\|_{\frac{N}{N-1}} \leq \prod_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_1^{\frac{1}{N}}.$$

(See [14, (19), P. 280].) Let $m \geq 1$. Applying (2.1) to $|u|^{m-1}u$, we get

$$\|u\|_{\frac{mN}{N-1}}^m \leq m \prod_{i=1}^N \left\| |u|^{m-1} \frac{\partial u}{\partial x_i} \right\|_1^{\frac{1}{N}} \leq C\|\nabla u\|_N \|u\|_{\frac{(m-1)N}{N-1}}^{m-1}.$$

By the Young inequality, it follows that

$$(2.2) \quad \|u\|_{\frac{mN}{N-1}} \leq C(\|u\|_{\frac{(m-1)N}{N-1}} + \|\nabla u\|_N) \quad \text{for any } m \geq 1.$$

In (2.2), we first choose $\frac{(m-1)N}{N-1} = \frac{pN}{N-p}$, that is, $m = \frac{(N-1)p}{N-p} + 1$. Writing $p^* = \frac{pN}{N-p}$ for simplicity, one has $m = \frac{N-1}{N}p^* + 1$ and $\frac{mN}{N-1} = p^* + \frac{N}{N-1}$. Thus from (2.2), we obtain

$$\|u\|_{p^* + \frac{N}{N-1}} \leq C(\|u\|_{p^*} + \|\nabla u\|_N) \leq C(\|\nabla u\|_p + \|\nabla u\|_N).$$

Iterating this procedure with $m = \frac{N-1}{N}p^* + j$ for $j \in \mathbb{N}$, and applying the interpolation inequality, one gets

$$\|u\|_r \leq C(\|\nabla u\|_p + \|\nabla u\|_N) \quad \text{for all } u \in C_0^\infty(\mathbb{R}^N) \text{ and } r \in [p^*, +\infty).$$

This completes the proof by a density argument.

CASE $1 < p < N < q$.

We argue as in [20]. Let $u \in C_0^\infty(\mathbb{R}^N)$, $x \in \mathbb{R}^N$ and Q be an open cube, containing x , whose sides -of length 1- are parallel to the coordinate axes. Going back to the proof of the Morrey inequality, we have

$$|\bar{u} - u(x)| \leq \frac{q}{q-N} \|\nabla u\|_{L^q(Q)},$$

where $\bar{u} = \frac{1}{|Q|} \int_Q u(x) dx$. (See [14, (27), P. 283] for the proof.) By the Hölder inequality, we arrive at

$$\begin{aligned} |u(x)| &\leq \left| \frac{1}{|Q|} \int_Q u(x) dx \right| + C \|\nabla u\|_{L^q(Q)} \leq C \|u\|_{L^{p^*}(Q)} + C \|\nabla u\|_{L^q(Q)} \\ &\leq C(\|u\|_{L^{p^*}(\mathbb{R}^N)} + \|\nabla u\|_{L^q(\mathbb{R}^N)}) \leq C(\|\nabla u\|_{L^p(\mathbb{R}^N)} + \|\nabla u\|_{L^q(\mathbb{R}^N)}), \end{aligned}$$

from which we deduce that

$$\|u\|_\infty \leq C(\|\nabla u\|_p + \|\nabla u\|_q).$$

Again, we conclude by a density argument. \square

Remark 2.2. By Theorem 2.1, for any $1 < p < q$ with $p < N$, according with the definitions of p^* and q^* given in the Introduction, one has

$$(2.3) \quad \mathcal{H}_0^{p,q} \hookrightarrow L^r(\mathbb{R}^N) \text{ for any } r \in [p^*, q^*].$$

Moreover the following property will be useful later

Proposition 2.3. Let $1 < p < q$ with $p < N$. Then, for any $u \in \mathcal{H}_0^{p,q}$ and $r \in [p^*, q^*]$, we have

$$(2.4) \quad \|u\|_r^r \leq C(\|\nabla u\|_p^r + \|\nabla u\|_p^{p^*} + \|\nabla u\|_q^{q^*}).$$

Proof. Let $u \in \mathcal{H}_0^{p,q}$ and $r \in [p^*, q^*]$. By Theorem 2.1, the interpolation inequality and the Young inequality, we get

$$\begin{aligned} \|u\|_r^r &\leq \|u\|_{p^*}^{\theta p^*} \|u\|_{q^*}^{(1-\theta)q^*} \leq C \|\nabla u\|_p^{\theta p^*} (\|\nabla u\|_p + \|\nabla u\|_q)^{(1-\theta)q^*} \\ &\leq C \|\nabla u\|_p^{\theta p^*} (\|\nabla u\|_p^{(1-\theta)q^*} + \|\nabla u\|_q^{(1-\theta)q^*}) \\ &\leq C(\|\nabla u\|_p^r + \|\nabla u\|_p^{p^*} + \|\nabla u\|_q^{q^*}). \end{aligned}$$

Here $\theta \in [0, 1]$ is a constant chosen so that $r = \theta p^* + (1 - \theta)q^*$. \square

Let us define the functional $I : \mathcal{H}_0^{p,q} \rightarrow \mathbb{R}$ by

$$I(u) = \frac{1}{p} \|\nabla u\|_p^p + \frac{1}{q} \|\nabla u\|_q^q - \int_{\mathbb{R}^N} G(u) dx.$$

By hypotheses (g1)-(g3), we can see that I is well-defined and of class C^1 on $\mathcal{H}_0^{p,q}$. Moreover any critical points of I are solutions of (P).

Next we truncate and decompose the nonlinear term g similarly as in [11]. Let us put

$$s_0 := \min\{s \in [\zeta, +\infty) \mid g(s) = 0\}$$

and $s_0 = +\infty$ if $g(s) \neq 0$ for all $s \geq \zeta$. We define $\tilde{g} : \mathbb{R}_+ \rightarrow \mathbb{R}$ by

$$\tilde{g}(s) = \begin{cases} g(s) & \text{on } [0, s_0], \\ 0 & \text{on } (s_0, +\infty). \end{cases}$$

By the maximum principle, any positive solutions of (\mathcal{P}) with \tilde{g} satisfy the original problem (\mathcal{P}) . Thus we may replace g by \tilde{g} in (\mathcal{P}) . Hereafter we write $g = \tilde{g}$ for simplicity. For $s \geq 0$, we set

$$g_1(s) := g_+(s), \quad \text{and} \quad g_2(s) := g_1(s) - g(s).$$

Then by (g2) and (g3), one has

$$(2.5) \quad \lim_{s \rightarrow 0} \frac{g_1(s)}{s^{p^*-1}} = 0, \quad \lim_{s \rightarrow +\infty} \frac{g_1(s)}{s^{q^*-1}} = 0.$$

Thus, for $s \geq 0$, we have from (2.5) that

$$(2.6) \quad 0 \leq g_1(s) \leq C(s^{p^*-1} + s^{q^*-1}),$$

$$(2.7) \quad 0 \leq g_2(s).$$

Hence, denoting $G_i(t) = \int_0^t g_i(s) ds$ for $i = 1, 2$, we get

$$(2.8) \quad G_2(s) \geq 0 \text{ for all } s \in \mathbb{R},$$

and

$$(2.9) \quad 0 \leq G_1(s) \leq C(|s|^{p^*} + |s|^{q^*}) \text{ for all } s \in \mathbb{R}.$$

2.2. Existence of a positive solution of (\mathcal{P}) .

In all this section, we assume (g1)-(g4) and prove Theorem 1.1. To this end, we consider the following auxiliary problem:

$$(2.10) \quad -\Delta_p u - \Delta_q u + g_2(u) = \lambda g_1(u) \text{ in } \mathbb{R}^N$$

for λ close to 1. Our strategy is to find a solution of (2.10) and pass a limit $\lambda \nearrow 1$. We define the functional $I_\lambda : \mathcal{H}_0^{p,q} \rightarrow \mathbb{R}$ by

$$I_\lambda(u) = \frac{1}{p} \|\nabla u\|_p^p + \frac{1}{q} \|\nabla u\|_q^q + \int_{\mathbb{R}^N} G_2(u) dx - \lambda \int_{\mathbb{R}^N} G_1(u) dx.$$

In order to find a non-trivial critical point of I_λ , we apply a slightly modified version of the Monotonicity trick due to [24] (see also [7]).

Proposition 2.4 (Monotonicity trick). *Let $(X, \|\cdot\|)$ be a Banach space and $J \subset \mathbb{R}^+$ an interval. Consider a family of C^1 functionals I_λ on X defined by*

$$I_\lambda(u) = A(u) - \lambda B(u) \text{ for } \lambda \in J,$$

with B non-negative and either $A(u) \rightarrow +\infty$ or $B(u) \rightarrow +\infty$ as $\|u\| \rightarrow +\infty$ and such that $I_\lambda(0) = 0$. For any $\lambda \in J$, we set

$$(2.11) \quad \Gamma_\lambda := \{\gamma \in C([0, 1], X) \mid \gamma(0) = 0, I_\lambda(\gamma(1)) < 0\}.$$

Assume that for every $\lambda \in J$, the set Γ_λ is non-empty and

$$(2.12) \quad c_\lambda := \inf_{\gamma \in \Gamma_\lambda} \max_{t \in [0, 1]} I_\lambda(\gamma(t)) > 0.$$

Then for almost every $\lambda \in J$, there is a sequence $\{v_n\} \subset X$ such that

- (i) $\{v_n\}$ is bounded in X ;
- (ii) $I_\lambda(v_n) \rightarrow c_\lambda$;
- (iii) $(I_\lambda)'(v_n) \rightarrow 0$ in the dual space X^{-1} of X .

In our case, we set $X = \mathcal{H}_{0,\text{rad}}^{p,q}$, where

$$\mathcal{H}_{0,\text{rad}}^{p,q} = \{u \in \mathcal{H}_0^{p,q} \mid u \text{ is radially symmetric}\},$$

and

$$A(u) = \frac{1}{p} \|\nabla u\|_p^p + \frac{1}{q} \|\nabla u\|_q^q + \int_{\mathbb{R}^N} G_2(u) dx,$$

$$B(u) = \int_{\mathbb{R}^N} G_1(u) dx.$$

To apply Proposition 2.4, we begin with the following lemma.

Lemma 2.5. *There exists $\lambda_0 \in (0, 1)$ such that the set Γ_λ defined in (2.11) is non-empty for every $\lambda \in J = [\lambda_0, 1]$.*

Proof. First by (g4), there exists $z \in \mathcal{H}_{0,\text{rad}}^{p,q}$ such that $\int_{\mathbb{R}^N} G(z) dx > 0$. (See [11, Proof of Theorem 2, P. 325].) Since $G(s) = G_1(s) - G_2(s)$, there exists $0 < \lambda_0 < 1$ such that

$$(2.13) \quad \lambda_0 \int_{\mathbb{R}^N} G_1(z) dx - \int_{\mathbb{R}^N} G_2(z) dx > 0.$$

Let $\lambda \in J = [\lambda_0, 1]$ and $t > 0$. We compute $I_\lambda(z(\frac{\cdot}{t}))$. From (2.13), one has

$$\begin{aligned} I_\lambda\left(z\left(\frac{\cdot}{t}\right)\right) &= \frac{t^{N-p}}{p} \|\nabla z\|_p^p + \frac{t^{N-q}}{q} \|\nabla z\|_q^q + t^N \int_{\mathbb{R}^N} G_2(z) dx - \lambda t^N \int_{\mathbb{R}^N} G_1(z) dx \\ &\leq \frac{t^{N-p}}{p} \|\nabla z\|_p^p dx + \frac{t^{N-q}}{q} \|\nabla z\|_q^q dx - t^N \left(\lambda_0 \int_{\mathbb{R}^N} G_1(z) dx - \int_{\mathbb{R}^N} G_2(z) dx \right). \end{aligned}$$

Hence, we can choose $\tau > 1$ so that $I_\lambda(z(\frac{\cdot}{\tau})) < 0$ and consider a function $\gamma : [0, 1] \rightarrow \mathcal{H}_{0,\text{rad}}^{p,q}$ which is defined by

$$\gamma(t) = \begin{cases} 2tz\left(\frac{2\cdot}{\tau}\right) & \text{if } t \in [0, 1/2], \\ z\left(\frac{\cdot}{t\tau}\right) & \text{if } t \in [1/2, 1]. \end{cases}$$

Then it follows that $\gamma \in \Gamma_\lambda$ and hence the proof is complete. \square

Lemma 2.6. *For all $\lambda \in J = [\lambda_0, 1]$, the condition (2.12) holds.*

Proof. For any $u \in \mathcal{H}_{0,\text{rad}}^{p,q}$ and $\lambda \in J$, we have from (2.8) and (2.9) that

$$I_\lambda(u) \geq \frac{1}{p} \|\nabla u\|_p^p + \frac{1}{q} \|\nabla u\|_q^q - C\|u\|_{p^*}^{p^*} - C\|u\|_{q^*}^{q^*}.$$

Thus by applying (2.4) with $r = p^*$ and $r = q^*$ respectively, one gets

$$\begin{aligned} I_\lambda(u) &\geq \frac{1}{p} \|\nabla u\|_p^p + \frac{1}{q} \|\nabla u\|_q^q - C(\|\nabla u\|_p^{p^*} + \|\nabla u\|_q^{q^*}) - C(\|\nabla u\|_p^{p^*} + \|\nabla u\|_p^{q^*} + \|\nabla u\|_q^{q^*}) \\ &\geq \frac{1}{p} \|\nabla u\|_p^p \left(1 - C\|\nabla u\|_p^{p^*-p} - C\|\nabla u\|_p^{q^*-p}\right) + \frac{1}{q} \|\nabla u\|_q^q \left(1 - C\|\nabla u\|_q^{q^*-q}\right). \end{aligned}$$

Let $u \in \mathcal{H}_{0,\text{rad}}^{p,q}$ be such that $\|u\|_{\mathcal{H}_0^{p,q}} = \|\nabla u\|_p + \|\nabla u\|_q = \rho < 1$. Since $q^* > p^* > p$ and $q^* > q$, if $\rho > 0$ is sufficiently small, it follows that

$$1 - C\|\nabla u\|_p^{p^*-p} - C\|\nabla u\|_p^{q^*-p} \geq \frac{1}{2}, \quad 1 - C\|\nabla u\|_q^{q^*-q} \geq \frac{1}{2}.$$

Then from $p < q$ and $\|\nabla u\|_p \leq \|u\|_{\mathcal{H}_0^{p,q}} < 1$, we obtain

$$I_\lambda(u) \geq \frac{1}{2p} \|\nabla u\|_p^p + \frac{1}{2q} \|\nabla u\|_q^q \geq \frac{1}{2p} \|\nabla u\|_p^q + \frac{1}{2q} \|\nabla u\|_q^q \geq C\|u\|_{\mathcal{H}_0^{p,q}}^q.$$

Thus there exists $\delta > 0$ such that $I_\lambda(u) \geq \delta$ for all $u \in \mathcal{H}_{0,\text{rad}}^{p,q}$ with $\|u\|_{\mathcal{H}_0^{p,q}} \leq \rho$.

Now we fix $\lambda \in J$ and $\gamma \in \Gamma_\lambda$. Since $\gamma(0) = 0 \neq \gamma(1)$ and $I_\lambda(\gamma(1)) < 0$, it follows that

$\|\gamma(1)\|_{\mathcal{H}_0^{p,q}} > \rho$. By continuity, we deduce that there exists $t_\gamma \in (0, 1)$ such that $\|\gamma(t_\gamma)\| = \rho$. Thus for any $\lambda \in J$, we obtain

$$\alpha \leq \inf_{\gamma \in \Gamma} I_\lambda(\gamma(t_\lambda)) \leq c_\lambda.$$

This completes the proof. \square

By Lemmas 2.5 and 2.6, we can apply Proposition 2.4 to obtain a bounded Palais-Smale sequence $\{u_n^\lambda\} \subset \mathcal{H}_{0,\text{rad}}^{p,q}$ of I_λ for almost every $\lambda \in J$, that is,

$$I_\lambda(u_n^\lambda) \rightarrow c_\lambda, \quad I'_\lambda(u_n^\lambda) \rightarrow 0 \quad \text{and} \quad \{u_n^\lambda\} \text{ is bounded in } \mathcal{H}_0^{p,q}$$

Hence, passing to a subsequence, there exists $u_\lambda \in \mathcal{H}_{0,\text{rad}}^{p,q}$ such that

$$(2.14) \quad \begin{aligned} u_n^\lambda &\rightharpoonup u_\lambda \text{ in } \mathcal{H}_0^{p,q}, \quad \text{as } n \rightarrow +\infty \\ u_n^\lambda(x) &\rightarrow u_\lambda(x) \text{ a.e. } x \in \mathbb{R}^N, \quad \text{as } n \rightarrow +\infty. \end{aligned}$$

Lemma 2.7. *The weak limit u_λ satisfies*

$$u_\lambda \neq 0, \quad I'_\lambda(u_\lambda) = 0 \quad \text{and} \quad I_\lambda(u_\lambda) \leq c_\lambda.$$

Proof. First we claim that

$$(2.15) \quad \int_{\mathbb{R}^N} G_1(u_n^\lambda) dx \rightarrow \int_{\mathbb{R}^N} G_1(u_\lambda) dx,$$

$$(2.16) \quad \int_{\mathbb{R}^N} g_1(u_n^\lambda) u_n^\lambda dx \rightarrow \int_{\mathbb{R}^N} g_1(u_\lambda) u_\lambda dx,$$

and, for any $\varphi \in C_0^\infty(\mathbb{R}^N)$ and for $i = 1, 2$,

$$(2.17) \quad \int_{\mathbb{R}^N} g_i(u_n^\lambda) \varphi dx \rightarrow \int_{\mathbb{R}^N} g_i(u_\lambda) \varphi dx.$$

To this end, we apply the compactness lemma due to Strauss. (See Lemma A.2 below.)

Let $Q(s) = |s|^{p^*} + |s|^{q^*}$. Then from (2.5), it follows that $\frac{G_1(s)}{Q(s)} \rightarrow 0$ as $s \rightarrow 0$ and $s \rightarrow \infty$. Moreover from (2.14), we also have $G_1(u_n^\lambda(x)) \rightarrow G_1(u_\lambda(x))$ a.e. $x \in \mathbb{R}^N$ and, by (2.3)

$$\sup_{n \in \mathbb{N}} \int_{\mathbb{R}^N} Q(u_n^\lambda) dx \leq C \sup_{n \in \mathbb{N}} \left(\|u_n^\lambda\|_{\mathcal{H}_0^{p,q}}^{p^*} + \|u_n^\lambda\|_{\mathcal{H}_0^{p,q}}^{q^*} \right) < +\infty.$$

Finally since $u_n^\lambda \in \mathcal{H}_{0,\text{rad}}^{p,q} \subset D_{\text{rad}}^{1,p}(\mathbb{R}^N)$, we have, by the radial lemma (see Lemma A.1 below), that $u_n^\lambda(x) \rightarrow 0$ as $|x| \rightarrow +\infty$ uniformly in $n \in \mathbb{N}$. Thus all assumptions in Lemma A.2 are satisfied. Then it follows that $G_1(u_n^\lambda) \rightarrow G_1(u_\lambda)$ in $L^1(\mathbb{R}^N)$ and hence (2.15) holds. Arguing similarly, one can show that (2.16) and (2.17).

Now from (2.17), $I'_\lambda(u_n^\lambda) \rightarrow 0$ and $u_n^\lambda \rightharpoonup u_\lambda$ in $\mathcal{H}_0^{p,q}$, one has $I'_\lambda(u_\lambda) = 0$. To prove $u_\lambda \neq 0$, we suppose by contradiction that $u_\lambda = 0$. Since $I'_\lambda(u_n^\lambda) \rightarrow 0$, we have by the boundedness of $\{u_n^\lambda\}$ in $\mathcal{H}_0^{p,q}$ that

$$\|\nabla u_n^\lambda\|_p^p + \|\nabla u_n^\lambda\|_q^q + \int_{\mathbb{R}^N} g_2(u_n^\lambda) u_n^\lambda dx = \lambda \int_{\mathbb{R}^N} g_1(u_n^\lambda) u_n^\lambda dx + o(1).$$

Then from (2.7) and (2.16), it follows that $\|u_n^\lambda\|_{\mathcal{H}_0^{p,q}} \rightarrow 0$, which contradicts $I_\lambda(u_n^\lambda) \rightarrow c_\lambda > 0$. Finally we show that $I_\lambda(u_\lambda) \leq c_\lambda$. By (2.14) and Fatou's lemma, one has

$$\int_{\mathbb{R}^N} G_2(u_\lambda) dx \leq \liminf_{n \rightarrow +\infty} \int_{\mathbb{R}^N} G_2(u_n^\lambda) dx.$$

By the weakly lower semi-continuity of $\|\cdot\|_{\mathcal{H}_0^{p,q}}$ and from (2.15), we obtain $I_\lambda(u_\lambda) \leq c_\lambda$. This completes the proof. \square

Lemma 2.7 implies that, for almost every $\lambda \in J$, u_λ is a non-trivial solution of (3.1). In order to obtain a non-trivial solution of the original problem (\mathcal{P}), we next consider a sequence of such $\{\lambda_n\}$ such that $\lambda_n \nearrow 1$ as $n \rightarrow +\infty$. Then by Proposition 2.4 and Lemma 2.7, there exists $\{v_n\} \subset \mathcal{H}_{0,\text{rad}}^{p,q} \setminus \{0\}$ such that

$$(2.18) \quad I'_{\lambda_n}(v_n) = 0, \quad I_{\lambda_n}(v_n) \leq c_{\lambda_n}.$$

Then we claim the following lemma.

Lemma 2.8. *The sequence $\{v_n\}$ is bounded in $\mathcal{H}_0^{p,q}$.*

Proof. First we observe from $I'_{\lambda_n}(v_n) = 0$ that v_n satisfies

$$-\Delta_p v_n - \Delta_q v_n + g_2(v_n) - \lambda_n g_1(v_n) = 0 \quad \text{in } \mathbb{R}^N,$$

in the weak sense. Next we claim that v_n satisfies the following Pohozaev identity:

$$(2.19) \quad \frac{N-p}{p} \|\nabla v_n\|_p^p + \frac{N-q}{q} \|\nabla v_n\|_q^q + N \int_{\mathbb{R}^N} G_2(v_n) dx - N \lambda_n \int_{\mathbb{R}^N} G_1(v_n) dx = 0.$$

To this aim, we argue as in [27]. First adapting the Moser type iteration as in [22], one can show that $v_n \in C_{\text{loc}}^{1,\sigma}(\mathbb{R}^N)$ for some $\sigma \in (0, 1)$.¹ Next since $v_n \in C_{\text{loc}}^{1,\sigma}(\mathbb{R}^N)$ and the function $\mathcal{L}(\xi) = \frac{1}{p}|\xi|^p + \frac{1}{q}|\xi|^q$ associated with the differential operator in (\mathcal{P}) is convex, we can apply the Pohozaev identity for C^1 solutions due to [16, Lemma 1] by choosing $h(x) = h_k(x) = H(x/k)x \in C_0^1(B_{2k}(0), \mathbb{R}^N)$ for $k \in \mathbb{N}$, where $H \in C_0^1(\mathbb{R}^N)$ is such that $H(x) = 1$ on $|x| \leq 1$ and $H(x) = 0$ for $|x| \geq 2$. Letting $k \rightarrow +\infty$ and taking into account that $|\nabla v_n|^p, |\nabla v_n|^q, G_1(v_n), G_2(v_n) \in L^1(\mathbb{R}^N)$, we obtain (2.19) as claimed.

Now from (2.18), we have

$$(2.20) \quad I_{\lambda_n}(v_n) = \frac{1}{p} \|\nabla v_n\|_p^p + \frac{1}{q} \|\nabla v_n\|_q^q + \int_{\mathbb{R}^N} G_2(v_n) dx - \lambda_n \int_{\mathbb{R}^N} G_1(v_n) dx \leq c_{\lambda_n}.$$

Hence from (2.19), (2.20) and the monotonicity of c_λ with respect to λ , it follows that

$$\|\nabla v_n\|_p^p + \|\nabla v_n\|_q^q \leq N c_{\lambda_n} \leq N c_{\lambda_0}$$

from which we conclude that the assertion holds. \square

We can now prove our first main result.

Proof of Theorem 1.1. By Lemma 2.8, up to a subsequence, we may assume that there exists $v \in \mathcal{H}_{0,\text{rad}}^{p,q}$ such that $v_n \rightharpoonup v$ in $\mathcal{H}_0^{p,q}$. Our goal is to show that v is a nontrivial critical point of I . First we prove that $I'(v) = 0$. To this aim, we observe from $I'_{\lambda_n}(v_n) = 0$ that

$$I'(v_n) = I'_{\lambda_n}(v_n) + (\lambda_n - 1)g_1(v_n) = (\lambda_n - 1)g_1(v_n).$$

Moreover arguing similarly as the proof of (2.17), one has

$$\int_{\mathbb{R}^N} g_1(v_n) \varphi dx \rightarrow \int_{\mathbb{R}^N} g_1(v) \varphi dx \quad \text{for any } \varphi \in C_0^\infty(\mathbb{R}^N).$$

This implies that $(\lambda_n - 1)g_1(v_n) = o(1)$ and hence $\{v_n\}$ is a Palais-Smale sequence for the functional I . Using the compactness lemma A.2 again, we can see that $I'(v) = 0$.

To conclude the proof, we claim that $v \neq 0$. Now from (2.6), (2.7) and $I'_{\lambda_n}(v_n) = 0$, we have

$$(2.21) \quad \begin{aligned} \|\nabla v_n\|_p^p + \|\nabla v_n\|_q^q &\leq \|\nabla v_n\|_p^p + \|\nabla v_n\|_q^q + \int_{\mathbb{R}^N} g_2(v_n) v_n dx \\ &= \lambda_n \int_{\mathbb{R}^N} g_1(v_n) v_n dx \leq C(\|v_n\|_{p^*}^p + \|v_n\|_{q^*}^q). \end{aligned}$$

¹First we notice that the argument used in the proof of Theorem 2 in [22] only requires that $v_n \in W_{\text{loc}}^{1,q}(\mathbb{R}^N)$ because they adopt a cut-off function to obtain desired estimates. Thus if $q < N$, we can apply Theorems 1-2 in [22] directly. If $q > N$, we already have $v_n \in L^\infty(\mathbb{R}^N)$ so that we can use Theorem 1 in [22]. Finally when $q = N$, we have to show that $v_n \in L^\infty(\mathbb{R}^N)$ first. But checking the proof of Theorem 2 in [22] carefully, the argument of the proof works even for $q = N$. Then we can apply Theorem 1 of [22] to conclude that $v_n \in C_{\text{loc}}^{1,\sigma}(\mathbb{R}^N)$.

Next we claim that

$$(2.22) \quad \liminf_{n \rightarrow +\infty} \|v_n\|_{\mathcal{H}_0^{p,q}} > 0.$$

Suppose by contradiction that $v_n \rightarrow 0$ in $\mathcal{H}_0^{p,q}$. Now from (2.4) and (2.21), one has

$$\|\nabla v_n\|_p^p + \|\nabla v_n\|_q^q \leq C \left(\|\nabla v_n\|_p^{p^*} + \|\nabla v_n\|_q^{q^*} + \|\nabla v_n\|_q^{q^*} \right).$$

Since $\|v_n\|_{\mathcal{H}_0^{p,q}} = \|\nabla v_n\|_p + \|\nabla v_n\|_q \rightarrow 0$, $p < p^* < q^*$ and $q < q^*$, we may assume that

$$C(\|\nabla v_n\|_p^{p^*} + \|\nabla v_n\|_q^{q^*}) \leq \frac{1}{2} \|\nabla v_n\|_p^p, \quad C\|\nabla v_n\|_q^{q^*} \leq \frac{1}{2} \|\nabla v_n\|_q^q,$$

from which we reach a contradiction.

By the compactness lemma A.2, one can show that

$$\int_{\mathbb{R}^N} g_1(v_n) v_n dx \rightarrow \int_{\mathbb{R}^N} g_1(v) v dx \text{ as } n \rightarrow +\infty.$$

Then from (2.21) and (2.22), we obtain

$$0 < \liminf_{n \rightarrow +\infty} (\|\nabla v_n\|_p^p + \|\nabla v_n\|_q^q) \leq \liminf_{n \rightarrow +\infty} \lambda_n \int_{\mathbb{R}^N} g_1(v_n) v_n dx = \int_{\mathbb{R}^N} g_1(v) v dx.$$

This implies that $v \neq 0$ and so we obtain the existence of a nontrivial solution of (P). Applying the Moser type iteration as in [22], one has $v \in C_{loc}^{1,\sigma}(\mathbb{R}^N)$ for some $\sigma \in (0, 1)$. Then from $g(s) \equiv 0$ for $s \leq 0$ and the Harnack inequality due to [33], it follows that $v > 0$ in \mathbb{R}^N . Finally by the radial lemma A.1, $v \in L^\infty(\mathbb{R}^N)$ and $v(x) \rightarrow 0$ as $|x| \rightarrow \infty$. This completes the proof of Theorem 1.1. \square

We conclude this section by showing the existence of a radial ground state solution of (P). Let us define by $\mathcal{S}_{0,\text{rad}}$ the set of the nontrivial radial solutions of (P), namely

$$\mathcal{S}_{0,\text{rad}} = \{u \in \mathcal{H}_{0,\text{rad}}^{p,q} \setminus \{0\} \mid I'(u) = 0\}.$$

By Theorem 1.1, we know that $\mathcal{S}_{0,\text{rad}} \neq \emptyset$. Arguing as in the proof of Theorem 1.1, we have

$$(2.23) \quad \inf_{u \in \mathcal{S}_{0,\text{rad}}} \|u\|_{\mathcal{H}_0^{p,q}} > 0.$$

In a similar argument as in the proof of Lemma 2.8, any $u \in \mathcal{S}_{0,\text{rad}}$ satisfies the following Pohozaev identity:

$$\frac{N-p}{pN} \|\nabla u\|_p^p + \frac{N-q}{qN} \|\nabla u\|_q^q = \int_{\mathbb{R}^N} G(u) dx.$$

Thus we infer that

$$(2.24) \quad I(u) = \frac{1}{N} (\|\nabla u\|_p^p + \|\nabla u\|_q^q).$$

Combining (2.23) and (2.24), we have that

$$\sigma = \inf_{u \in \mathcal{S}_{0,\text{rad}}} I(u) > 0.$$

Theorem 2.9. *Assume (g1)-(g4). Then (P) has a radial ground state solution, namely there exists $\bar{u} \in \mathcal{S}_{0,\text{rad}}$ such that*

$$I(\bar{u}) = \min_{u \in \mathcal{S}_{0,\text{rad}}} I(u).$$

Proof. Let $\{u_n\} \subset \mathcal{S}_{0,\text{rad}}$ be a minimizing sequence. Since

$$I(u_n) = \frac{1}{N} (\|\nabla u_n\|_p^p + \|\nabla u_n\|_q^q) \rightarrow \sigma,$$

we infer that $\{u_n\}$ is bounded in $\mathcal{H}_{0,\text{rad}}^{p,q}$. Therefore there exists $\bar{u} \in \mathcal{H}_{0,\text{rad}}^{p,q}$ such that $u_n \rightharpoonup \bar{u}$ weakly in $\mathcal{H}_{0,\text{rad}}^{p,q}$.

Arguing as in the proof of Theorem 1.1, we have that $\bar{u} \in \mathcal{S}_{0,\text{rad}}$ and so we conclude observing that, by the weak lower semicontinuity of the norms,

$$\sigma \leq I(\bar{u}) = \frac{1}{N} (\|\nabla \bar{u}\|_p^p + \|\nabla \bar{u}\|_q^q) \leq \liminf_{n \rightarrow +\infty} \frac{1}{N} (\|\nabla u_n\|_p^p + \|\nabla u_n\|_q^q) = \liminf_{n \rightarrow +\infty} I(u_n) = \sigma.$$

This completes the proof. \square

Remark 2.10. In Theorem 2.9, we could only obtain the existence of a radial ground state solution. We expect that the existence of ground state solutions can be shown without restricting ourselves to the radial class. For this purpose, we have two possibilities.

One is to characterize the ground state solution as a constraint minimizer of suitable functional. Then the result on the symmetry of constraint minimizers due to [25] enables us to conclude that any ground state solution is radially symmetric. For 0-th order problem (1.2), the ground state solution can be characterized as the following constraint minimizer:

$$\inf \left\{ \|\nabla u\|_2 \mid u \in H^1(\mathbb{R}^N), \int_{\mathbb{R}^N} G(u) dx = 1 \right\}.$$

However in order to characterize the ground state solution in this way, scaling property plays an essential role. Since scaling argument fails to work in our problem, we don't know whether the ground state solution of (P) can be characterized as a constraint minimizer of some suitable functional.

The other possibility is to apply the concentration compactness principle as in [6]. But in order to adopt their argument, we also need the characterization of the ground state solution.

It is also worth pointing out that, if the nonlinearity $g(s)$ is locally Lipschitz continuous for $s \geq 0$, we can apply the symmetry result due to [29] for the problem (P), to show that any non-negative decaying solution of class C^1 is radially symmetric.

Finally if we assume that $g(s)$ is odd as in [11], we are not able to say that any ground state solution of (P) is positive. This is because generically, the proof of the positivity of ground state solutions is based on the characterization by constraint minimization, which is not available for our problem.

3. THE POSITIVE MASS CASE

This section is devoted to the study of (P), in the positive mass case, namely when g satisfies (g2') instead of (g2). In this case, we work on the function space $\mathcal{H}^{p,q}$ which is given by $\mathcal{H}^{p,q} = \overline{C_0^\infty(\mathbb{R}^N)}^{\|\cdot\|_{\mathcal{H}^{p,q}}}$, where

$$\|u\|_{\mathcal{H}^{p,q}} := \|\nabla u\|_p + \|u\|_\ell + \|\nabla u\|_q.$$

If $\ell = p^*$, $\|\cdot\|_\ell$ can be dropped. For all $N \geq 3$, it follows that $\mathcal{H}^{p,q} \hookrightarrow \mathcal{H}_0^{p,q}$ and $\mathcal{H}^{p,q} \hookrightarrow L^\ell(\mathbb{R}^N)$. Thus by Theorem 2.1 and since $\ell \in [p, p^*]$, we have

$$(3.1) \quad \mathcal{H}^{p,q} \hookrightarrow L^r(\mathbb{R}^N) \quad \text{for all } r \in [\ell, q^*].$$

For $u \in \mathcal{H}^{p,q}$, we define the functional $I : \mathcal{H}^{p,q} \rightarrow \mathbb{R}$ associated with (P) by

$$I(u) = \frac{1}{p} \|\nabla u\|_p^p + \frac{1}{q} \|\nabla u\|_q^q - \int_{\mathbb{R}^N} G(u) dx.$$

By hypotheses (g1), (g2'), (g3) and from (3.1), we can see that I is well-defined and of class C^1 on $\mathcal{H}^{p,q}$ and its critical points are solutions of (P).

As done in Section 2, we truncate and decompose the nonlinear term g . Let

$$s_0 := \min\{s \in [\zeta, +\infty) \mid g(s) = 0\}$$

and $s_0 = +\infty$ if $g(s) \neq 0$ for all $s \geq \zeta$. We define $\tilde{g} : \mathbb{R}_+ \rightarrow \mathbb{R}$ by

$$\tilde{g}(s) = \begin{cases} g(s) & \text{on } [0, s_0], \\ 0 & \text{on } (s_0, +\infty). \end{cases}$$

Also in this case, by the maximum principle, any positive solutions of (\mathcal{P}) with \tilde{g} satisfy the original problem (\mathcal{P}) and thus we may replace g by \tilde{g} in (\mathcal{P}) . Hereafter we write $g = \tilde{g}$ for simplicity.

Next for $s \geq 0$, we set

$$g_1(s) := (g(s) + m_\ell s^{\ell-1})_+ \quad \text{and} \quad g_2(s) := g_1(s) - g(s).$$

Then one has $g_1(s) \geq 0$, $g_2(s) \geq 0$ for $s \geq 0$ and

$$(3.2) \quad \lim_{s \rightarrow 0} \frac{g_1(s)}{|s|^{\ell-1}} = 0,$$

$$(3.3) \quad \lim_{s \rightarrow +\infty} \frac{g_1(s)}{s^{q^*-1}} = 0,$$

$$(3.4) \quad g_2(s) \geq m_\ell s^{\ell-1} \quad \text{for all } s \geq 0.$$

From (3.2)-(3.4), for any $0 < \varepsilon < 1$, there exists $C_\varepsilon > 0$ such that

$$(3.5) \quad g_1(s) \leq C_\varepsilon |s|^{q^*-1} + \varepsilon g_2(s) \quad \text{for } s \geq 0.$$

Moreover we put $G_i(t) = \int_0^t g_i(s) ds$ for $i = 1, 2$. Then from (3.4) and (3.5), we also have

$$(3.6) \quad G_2(s) \geq \frac{m_\ell}{\ell} |s|^\ell \quad \text{for all } s \in \mathbb{R},$$

$$(3.7) \quad G_1(s) \leq \frac{C_\varepsilon}{q^*} |s|^{q^*} + \varepsilon G_2(s) \quad \text{for all } s \in \mathbb{R}.$$

We follow the same strategy as in the previous section and so we consider the following auxiliary problem:

$$(3.8) \quad -\Delta_p u - \Delta_q u + g_2(u) = \lambda g_1(u) \quad \text{in } \mathbb{R}^N$$

for λ close to 1. We define the functional $I_\lambda : \mathcal{H}^{p,q} \rightarrow \mathbb{R}$ by

$$I_\lambda(u) = \frac{1}{p} \|\nabla u\|_p^p + \frac{1}{q} \|\nabla u\|_q^q + \int_{\mathbb{R}^N} G_2(u) dx - \lambda \int_{\mathbb{R}^N} G_1(u) dx.$$

In order to find a non-trivial critical point of I_λ , we apply the Monotonicity trick (see Proposition 2.4), with $X = \mathcal{H}_{\text{rad}}^{p,q}$, where

$$\mathcal{H}_{\text{rad}}^{p,q} = \{u \in \mathcal{H}^{p,q} \mid u \text{ is radially symmetric}\},$$

and

$$\begin{aligned} A(u) &= \frac{1}{p} \|\nabla u\|_p^p + \frac{1}{q} \|\nabla u\|_q^q + \int_{\mathbb{R}^N} G_2(u) dx, \\ B(u) &= \int_{\mathbb{R}^N} G_1(u) dx. \end{aligned}$$

Now arguing as in Lemma 2.5, we infer the following.

Lemma 3.1. *There exists $\lambda_0 \in (0, 1)$ such that the set Γ_λ defined in (2.11) is non-empty for every $\lambda \in J = [\lambda_0, 1]$.*

Next we establish the following lemma by modifying the proof of Lemma 2.6.

Lemma 3.2. *For all $\lambda \in J = [\lambda_0, 1]$, the condition (2.12) holds.*

Proof. For any $u \in \mathcal{H}_{\text{rad}}^{p,q}$ and $\lambda \in J$, we have from (3.6) and (3.7), and later by (2.4), that

$$\begin{aligned} I_\lambda(u) &\geq \frac{1}{p} \|\nabla u\|_p^p + \frac{1}{q} \|\nabla u\|_q^q + \frac{m_\ell(1-\varepsilon)}{\ell} \|u\|_\ell^\ell - \frac{C_\varepsilon}{q^*} \|u\|_{q^*}^{q^*} \\ &\geq \frac{1}{p} \|\nabla u\|_p^p \left(1 - C \|\nabla u\|_p^{p^*-p} - C \|\nabla u\|_p^{q^*-p}\right) + \frac{1}{q} \|\nabla u\|_q^q \left(1 - C \|\nabla u\|_q^{q^*-q}\right) + \frac{m_\ell(1-\varepsilon)}{\ell} \|u\|_\ell^\ell. \end{aligned}$$

Let $u \in \mathcal{H}_{\text{rad}}^{p,q}$ such that $\|u\|_{\mathcal{H}^{p,q}} = \|\nabla u\|_p + \|u\|_\ell + \|\nabla u\|_q = \rho < 1$. Since $q^* > p^* > p$ and $q^* > q$, if $\rho > 0$ is sufficiently small, it follows that

$$1 - C\|\nabla u\|_p^{p^*-p} - C\|\nabla u\|_p^{q^*-p} \geq \frac{1}{2}, \quad 1 - C\|\nabla u\|_q^{q^*-q} \geq \frac{1}{2}.$$

Then from $p \leq \ell$ and $p < q$, we get

$$I_\lambda(u) \geq C\|u\|_{\mathcal{H}^{p,q}}^{\bar{\ell}},$$

where $\bar{\ell} = \max\{\ell, q\}$. Therefore there exists $\delta > 0$ such that $I_\lambda(u) \geq \delta$ for all $u \in \mathcal{H}_{\text{rad}}^{p,q}$ with $\|u\|_{\mathcal{H}^{p,q}} \leq \rho$. The conclusion follows as in Lemma 2.6. \square

By Lemmas 3.1 and 3.2, we can apply Proposition 2.4 to obtain a bounded Palais-Smale sequence $\{u_n^\lambda\} \subset \mathcal{H}_{\text{rad}}^{p,q}$ of I_λ , for almost every $\lambda \in J$, that is,

$$I_\lambda(u_n^\lambda) \rightarrow c_\lambda, \quad I'_\lambda(u_n^\lambda) \rightarrow 0 \text{ and } \{u_n^\lambda\} \text{ is bounded in } \mathcal{H}^{p,q}$$

Hence, passing to a subsequence, there exists $u_\lambda \in \mathcal{H}_{\text{rad}}^{p,q}$ such that

$$\begin{aligned} u_n^\lambda &\rightharpoonup u_\lambda \text{ in } \mathcal{H}^{p,q}, \text{ as } n \rightarrow +\infty, \\ u_n^\lambda(x) &\rightarrow u_\lambda(x) \text{ a.e. } x \in \mathbb{R}^N, \text{ as } n \rightarrow +\infty. \end{aligned}$$

Lemma 3.3. *The weak limit u_λ satisfies*

$$u_\lambda \neq 0, \quad I'_\lambda(u_\lambda) = 0 \quad \text{and} \quad I_\lambda(u_\lambda) \leq c_\lambda.$$

Proof. The proof is almost same as that of Lemma 2.7. The only difference is the choice of $Q(s)$ to apply the Strauss's compactness lemma. Indeed in the positive mass case, putting $Q(s) = |s|^\ell + |s|^{q^*}$, one has from (3.2) and (3.3) that $\frac{G_1(s)}{Q(s)} \rightarrow 0$ as $s \rightarrow 0$ and $s \rightarrow \infty$. The rest of the proof can be done in a similar way as Lemma 2.7. \square

Lemma 3.3 implies that, for almost every $\lambda \in J$, u_λ is a non-trivial solution of (3.8). In order to obtain a non-trivial solution of the original problem (P), we next consider a sequence $\{\lambda_n\}$ such that $\lambda_n \nearrow 1$ as $n \rightarrow +\infty$. Then by Proposition 2.4 and Lemma 3.3, there exists $\{v_n\} \subset \mathcal{H}_{\text{rad}}^{p,q} \setminus \{0\}$ such that

$$(3.9) \quad I'_{\lambda_n}(v_n) = 0, \quad I_{\lambda_n}(v_n) \leq c_{\lambda_n}.$$

Then we claim the following lemma.

Lemma 3.4. *The sequence $\{v_n\}$ is bounded in $\mathcal{H}^{p,q}$.*

Proof. The conclusion follows by the same argument as in the proof of Lemma 2.8. But in the positive mass case, we further need a bound for the L^ℓ -norm of $\{v_n\}$.

Now since $I'_{\lambda_n}(v_n) = 0$, it follows that v_n satisfies

$$-\Delta_p v_n - \Delta_q v_n + g_2(v_n) - \lambda_n g_1(v_n) = 0 \quad \text{in } \mathbb{R}^N,$$

in the weak sense. Then as in the zero mass case, one can show that the following Pohozaev identity holds:

$$(3.10) \quad \frac{N-p}{p} \|\nabla v_n\|_p^p + \frac{N-q}{q} \|\nabla v_n\|_q^q + N \int_{\mathbb{R}^N} G_2(v_n) dx - N \lambda_n \int_{\mathbb{R}^N} G_1(v_n) dx = 0.$$

Moreover from (3.9), we also have

$$(3.11) \quad I_{\lambda_n}(v_n) = \frac{1}{p} \|\nabla v_n\|_p^p + \frac{1}{q} \|\nabla v_n\|_q^q + \int_{\mathbb{R}^N} G_2(v_n) dx - \lambda_n \int_{\mathbb{R}^N} G_1(v_n) dx \leq c_{\lambda_n},$$

$$(3.12) \quad I'_{\lambda_n}(v_n)[v_n] = \|\nabla v_n\|_p^p + \|\nabla v_n\|_q^q + \int_{\mathbb{R}^N} g_2(v_n) v_n dx - \lambda_n \int_{\mathbb{R}^N} g_1(v_n) v_n dx = 0.$$

From (3.10), (3.11) and the monotonicity of c_λ with respect to λ , it follows that

$$\|\nabla v_n\|_p^p + \|\nabla v_n\|_q^q \leq N c_{\lambda_n} \leq N c_{\lambda_0},$$

and hence

$$(3.13) \quad \|v_n\|_{\mathcal{H}_0^{p,q}} \leq C.$$

To conclude, we have to show that $\{v_n\}$ is bounded in $L^\ell(\mathbb{R}^N)$. By (3.4), (3.5) and (3.12), one has

$$\begin{aligned} 0 &= \|\nabla v_n\|_p^p + \|\nabla v_n\|_q^q + \int_{\mathbb{R}^N} g_2(v_n) v_n dx - \lambda_n \int_{\mathbb{R}^N} g_1(v_n) v_n dx \\ &\geq (1 - \varepsilon) m_\ell \|v_n\|_\ell^\ell - C_\varepsilon \|v_n\|_{q^*}^{q^*}. \end{aligned}$$

By Theorem 2.1 and (3.13), we get

$$\|v_n\|_\ell^\ell \leq C \|v_n\|_{q^*}^{q^*} \leq C \|v_n\|_{\mathcal{H}_0^{p,q}}^{q^*} \leq C.$$

This, together with (3.13), completes the proof. \square

Proof of Theorem 1.2. By Lemma 3.4, up to a subsequence, we may assume that there exists $v \in \mathcal{H}_{\text{rad}}^{p,q}$ such that $v_n \rightharpoonup v$ in $\mathcal{H}^{p,q}$. Arguing as in the proof of Theorem 1.1, we can see that v is a nontrivial critical point of I . Moreover by the regularity theory and the Harnack inequality, it follows that $v > 0$ in \mathbb{R}^N . \square

To introduce the existence of a ground state solution, let us define \mathcal{S}_{rad} the set of the nontrivial radial solutions of (\mathcal{P}) , namely

$$\mathcal{S}_{\text{rad}} = \{u \in \mathcal{H}_{\text{rad}}^{p,q} \setminus \{0\} \mid I'(u) = 0\}.$$

Arguing as Theorem 2.9, one can show the following result.

Theorem 3.5. *Assume (g1), (g2'), (g3) and (g4). Then (\mathcal{P}) has a radial ground state solution, namely there exists $\bar{u} \in \mathcal{S}_{\text{rad}}$ such that*

$$I(\bar{u}) = \min_{u \in \mathcal{S}_{\text{rad}}} I(u).$$

4. k -TH ORDER APPROXIMATED PROBLEM

In this section, we consider the approximation, at k -th order, of the Born-Infeld equation (1.2), namely we deal with the following problem:

$$(4.1) \quad \begin{cases} -\Delta u - \Delta_4 u \cdots - \Delta_{2k} u = h(u) & \text{in } \mathbb{R}^N, \\ u(x) \rightarrow 0 & \text{as } |x| \rightarrow +\infty, \end{cases}$$

where $N \geq 3$, $k \in \mathbb{N}$ with $k \geq 2$. Here we normalized the coefficients $\frac{(2k-3)!!}{(k-1)!} \beta^{k-1}$ in (1.3) because they are not essential for the existence of solutions. Moreover since we are interested in higher order approximation, we assume that

$$(4.2) \quad k \geq \max \left\{ \frac{N}{2}, \frac{N}{N-2} \right\}.$$

We impose the following assumptions on h .

(h1) $h \in C(\mathbb{R}_+, \mathbb{R})$ and $h(s) \equiv 0$ for $s \leq 0$.

(h2) Either (i) or (ii) is fulfilled:

(i) for all $\ell \in [2, \frac{2N}{N-2}]$, it holds

$$-\infty \leq \limsup_{s \rightarrow 0^+} \frac{h(s)}{s^{\ell-1}} \leq 0;$$

(ii) there exist $\ell \in [2, \frac{2N}{N-2}]$ and $m_\ell > 0$ such that

$$-\infty < \liminf_{s \rightarrow 0^+} \frac{h(s)}{s^{\ell-1}} \leq \limsup_{s \rightarrow 0^+} \frac{h(s)}{s^{\ell-1}} = -m_\ell.$$

(h3) There exists $\ell^* > 2k \geq \frac{2N}{N-2}$ such that

$$-\infty \leq \limsup_{s \rightarrow +\infty} \frac{h(s)}{s^{\ell^*-1}} \leq 0.$$

(h4) There exists $\zeta > 0$ such that $H(\zeta) = \int_0^\zeta h(s) ds > 0$.

In this setting, we have the following result.

Theorem 4.1. *Assume (4.2), (h1)-(h4). Then problem (4.1) has a solution which is positive and radially symmetric and belongs to $C_{\text{loc}}^{1,\sigma}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ class, for some $\sigma \in (0, 1)$. Furthermore, there exists a radial ground state solution of (4.1).*

As a special case, let us study the problem:

$$(4.3) \quad \begin{cases} -\Delta u - \beta \Delta_4 u - \frac{3}{2} \beta^2 \Delta_6 u \cdots - \frac{(2k-3)!!}{(k-1)!} \beta^{k-1} \Delta_{2k} u = |u|^{\alpha-1} u & \text{in } \mathbb{R}^N, \\ u(x) \rightarrow 0 & \text{as } |x| \rightarrow +\infty, \end{cases}$$

for $N \geq 3$, $\alpha > \frac{2N}{N-2}$ and $\beta > 0$. Under the assumption (4.2), we choose $\ell^* > \max\{\alpha, 2k\}$ arbitrarily and, by Theorem 4.1, we obtain the following result.

Corollary 4.2. *Assume (4.2) and let $\alpha > \frac{2N}{N-2}$ and $\beta > 0$ be arbitrarily given. Then the problem (4.3) has a positive radial solution as well as a radial ground state solution.*

We expect that under some smallness condition on β , a positive solution u_k of (4.3) converges to a positive solution of

$$(4.4) \quad \begin{cases} -\operatorname{div} \left(\frac{\nabla u}{\sqrt{1-2\beta|\nabla u|^2}} \right) = |u|^{\alpha-1} u & \text{in } \mathbb{R}^N, \\ u(x) \rightarrow 0 & \text{as } |x| \rightarrow +\infty, \end{cases}$$

as $k \rightarrow +\infty$ in a certain sense. But we postpone this question to a future work.

We also note that the problem (4.3) has no non-trivial C^1 solution if $1 < \alpha \leq \frac{2N}{N-2}$. Indeed for any non-trivial C^1 solution of (4.3), by adapting the argument in [16], one can prove the following two identities hold:

$$\begin{aligned} \|\nabla u\|_2^2 + \beta \|\nabla u\|_4^4 \cdots + \frac{(2k-3)!!}{(k-1)!} \beta^{k-1} \|\nabla u\|_{2k}^{2k} &= \|u\|_\alpha^\alpha \quad (\text{Nehari}) \\ \frac{N-2}{2} \|\nabla u\|_2^2 + \frac{N-4}{4} \beta \|\nabla u\|_4^4 \cdots + \frac{N-2k}{2k} \frac{(2k-3)!!}{(k-1)!} \beta^{k-1} \|\nabla u\|_{2k}^{2k} &= \frac{N}{\alpha} \|u\|_\alpha^\alpha \quad (\text{Pohozaev}). \end{aligned}$$

Substituting the first equation for the second one, we obtain

$$(4.5) \quad \left(\frac{N-2}{2} - \frac{N}{\alpha} \right) \|\nabla u\|_2^2 + \left(\frac{N-4}{4} - \frac{N}{\alpha} \right) \beta \|\nabla u\|_4^4 \\ + \cdots + \left(\frac{N-2k}{2k} - \frac{N}{\alpha} \right) \frac{(2k-3)!!}{(k-1)!} \beta^{k-1} \|\nabla u\|_{2k}^{2k} = 0.$$

If $1 < \alpha \leq \frac{2N}{N-2}$, it follows that $\frac{N}{\alpha} \geq \frac{N-2}{2}$ and hence

$$\frac{N-2j}{2j} - \frac{N}{\alpha} \leq \frac{N-2j}{2j} - \frac{N-2}{2} = -\frac{(j-1)N}{2j} \leq 0 \quad \text{for } j \geq 1.$$

This implies that all terms in the left hand side of (4.5) are non-negative, yielding that $\nabla u \equiv 0$ and hence $u \equiv 0$. We note that the non-existence of positive radial solutions of (4.4) for the case $1 < \alpha \leq \frac{2N}{N-2}$ has been obtained in [2] by the ODE technique.

The proof of Theorem 4.1 is almost the same as those of Theorems 1.1-1.2, 2.9 and 3.5. Here we consider the zero mass case (h2-i) and only give a sketch of the proof.

First we set a function space $\mathcal{H}_0^{2,2k}$ defined by $\mathcal{H}_0^{2,2k} = \overline{C_0^\infty(\mathbb{R}^N)}^{\|\cdot\|_{\mathcal{H}_0^{2,2k}}}$, where

$$\|u\|_{\mathcal{H}_0^{2,2k}} := \|\nabla u\|_2 + \|\nabla u\|_{2k}.$$

Since $2k \geq N$, it follows by Theorem 2.1 that $\mathcal{H}_0^{2,2k} \hookrightarrow L^r(\mathbb{R}^N)$ and

$$(4.6) \quad \|u\|_r \leq C(\|\nabla u\|_2 + \|\nabla u\|_{2k}) \text{ for any } u \in \mathcal{H}_0^{2,2k} \text{ and } r \in \left[\frac{2N}{N-2}, +\infty\right).$$

We define the functional $I_k : \mathcal{H}_0^{2,2k} \rightarrow \mathbb{R}$ by

$$I_k(u) = \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{4}\|\nabla u\|_4^4 \cdots + \frac{1}{2k}\|\nabla u\|_{2k}^{2k} - \int_{\mathbb{R}^N} H(u) dx,$$

which is well-defined and C^1 by (h1)-(h3). Moreover we truncate and decompose $h(s)$ as in Section 2.1. We apply the Monotonicity trick to $X = \mathcal{H}_{0,\text{rad}}^{2,2k}$, where

$$\mathcal{H}_{0,\text{rad}}^{2,2k} = \{u \in \mathcal{H}_0^{2,2k} \mid u \text{ is radially symmetric}\},$$

and consider the modified functional $I_{k,\lambda}$ which is given by

$$I_{k,\lambda}(u) = \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{4}\|\nabla u\|_4^4 \cdots + \frac{1}{2k}\|\nabla u\|_{2k}^{2k} + \int_{\mathbb{R}^N} H_2(u) dx - \lambda \int_{\mathbb{R}^N} H_1(u) dx$$

for $\lambda \in (0, 1]$.

The arguments from now on are similar to those of the previous sections and we omit the details.

APPENDIX A.

In this appendix, we collect some well known lemmas which we used in this paper.

Lemma A.1 (Radial Lemma, [11, 31]). *Suppose $1 < p < N$. Then there exists $C = C(N, p) > 0$ such that for any $u \in D_{\text{rad}}^{1,p}(\mathbb{R}^N)$,*

$$|u(x)| \leq C|x|^{-\frac{N-p}{p}} \|\nabla u\|_p.$$

Next we recall a variant of the Strauss' compactness lemma due to [7]. (See also [11, Theorem A.1], [30].) It will be a fundamental tool in our arguments.

Lemma A.2. *Let P and $Q : \mathbb{R} \rightarrow \mathbb{R}$ be two continuous functions satisfying*

$$\lim_{s \rightarrow \infty} \frac{P(s)}{Q(s)} = 0,$$

$\{v_n\}$, v and z be measurable functions from \mathbb{R}^N to \mathbb{R} , with z bounded, such that

$$\begin{aligned} \sup_{n \in \mathbb{N}} \int_{\mathbb{R}^N} |Q(v_n(x))z| dx &< +\infty, \\ P(v_n(x)) &\rightarrow v(x) \text{ a.e. in } \mathbb{R}^N. \end{aligned}$$

Then $\|(P(v_n) - v)z\|_{L^1(B)} \rightarrow 0$, for any bounded Borel set B .

Moreover, if we have also

$$\begin{aligned} \lim_{s \rightarrow 0} \frac{P(s)}{Q(s)} &= 0, \\ \lim_{x \rightarrow \infty} \sup_{n \in \mathbb{N}} |v_n(x)| &= 0, \end{aligned}$$

then $\|(P(v_n) - v)z\|_{L^1(\mathbb{R}^N)} \rightarrow 0$.

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REFERENCES

- [1] A. Azzollini, *Ground state solution for a problem with mean curvature operator in Minkowski space*, J. Funct. Anal. **266** (2014), 2086–2095.
- [2] A. Azzollini, *On a prescribed mean curvature equation in Lorentz-Minkowski space*, J. Math. Pures Appl. **106** (2016), 1122–1140.
- [3] A. Azzollini, P. d’Avenia, A. Pomponio, *Quasilinear elliptic equations in \mathbb{R}^N via variational methods and Orlicz-Sobolev embeddings*, Calc. Var. PDEs, **49**, (2014), 197–213.
- [4] A. Azzollini, A. Pomponio, *On a “zero mass” nonlinear Schrödinger equation*, Adv. Nonlinear Stud. **7**, (2007), 599–627.
- [5] A. Azzollini, A. Pomponio, *Compactness results and applications to some “zero mass” elliptic problems*, Nonlinear Anal. **69** (2008), 3559–3576.
- [6] A. Azzollini, A. Pomponio, *Ground state solutions for the nonlinear Schrödinger-Maxwell equations*, J. Math. Anal. Appl. **345** (2008), 90–108.
- [7] A. Azzollini, A. Pomponio, *On the Schrödinger equation in \mathbb{R}^N under the effect of a general nonlinear term*, Indiana Univ. Math. J. **58** (2009), 1361–1378.
- [8] M. Badiale, L. Pisani, S. Rolando, *Sum of weighted Lebesgue spaces and nonlinear elliptic equations*, NoDEA Nonlinear Diff. Equ. Appl. **18** (2011), 369–405.
- [9] R. Bartolo, A. M. Candela, A. Salvatore, *On a class of superlinear (p, q) -Laplacian type equations on \mathbb{R}^N* , J. Math. Anal. Appl. **438** (2016), 29–41.
- [10] V. Benci, A. M. Micheletti, *Solutions in exterior domains of null mass nonlinear field equations*, Adv. Nonlinear Stud. **6** (2006), 171–198.
- [11] H. Berestycki, P. L. Lions, *Nonlinear scalar fields equations, I. Existence of a ground state*, Arch. Rational Mech. Anal. **82** (1983), 313–345.
- [12] D. Bonheure, P. d’Avenia, A. Pomponio, *On the electrostatic Born-Infeld equation with extended charges*, Comm. Math. Phys. **346** (2016), 877–906.
- [13] D. Bonheure, C. De Coster, A. Derlet, *Infinitely many radial solutions of a mean curvature equation in Lorentz-Minkowski space*, Rend. Istit. Mat. Univ. Trieste. **44** (2012), 259–284.
- [14] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, New York, 2011.
- [15] M. F. Chaves, G. Ercole, O. H. Miyagaki, *Existence of a nontrivial solution for the (p, q) -Laplacian in \mathbb{R}^N without the Ambrosetti-Rabinowitz condition*, Nonlinear Anal. **114** (2015), 133–141.
- [16] M. Degiovanni, A. Musesti, M. Squassina, *On the regularity of solutions in the Pucci-Serrin identity*, Calc. Var. **18** (2003), 317–334.
- [17] J. M. Do Ó, E. Medeiros, *Remarks on least energy solutions for quasilinear elliptic problems in \mathbb{R}^N* , Elect. J. Diff. Eqns. **83** (2003), 1–14.
- [18] A. Ferrero, F. Gazzola, *On subcriticality assumptions for the existence of ground states of quasilinear elliptic equations*, Adv. Diff. Eqns. **8** (2003), 1081–1106.
- [19] G. M. Figueiredo, *Existence of positive solutions for a class of p & q elliptic problems with critical growth on \mathbb{R}^N* , J. Math. Anal. Appl. **378** (2011), 507–518.
- [20] D. Fortunato, L. Orsina, L. Pisani, *Born-Infeld type equations for electrostatic fields*, J. Math. Phys. **43** (2002), 5698–5706.
- [21] B. Franchi, E. Lanconelli, J. Serrin, *Existence and uniqueness of nonnegative solutions of quasilinear equations in \mathbb{R}^n* , Adv. Math. **118** (1996), 177–243.
- [22] C. He, G. Li, *The regularity of weak solutions to nonlinear scalar field elliptic equations containing p & q -Laplacians*, Anal. Acad. Sci. Fenn. **33** (2008), 337–371.
- [23] J. Hirata, N. Ikoma and K. Tanaka, *Nonlinear scalar field equations in \mathbb{R}^N : mountain pass and symmetric mountain pass approaches*, Top. Methods in Nonlinear Anal. **35** (2010), 253–276.
- [24] L. Jeanjean, *On the existence of bounded Palais-Smale sequences and application to a Landesman-Lazer-type problem set on \mathbb{R}* , Proc. Royal Soc. Edin. **129A** (1999), 787–809.
- [25] M. Mariş, *On the symmetry of minimizers*, Arch. Rat. Mech. Anal. **192** (2009), 311–330.
- [26] D. Mugnai, N. S. Papageorgiou, *Wang’s multiplicity result for superlinear (p, q) -equations without the Ambrosetti-Rabinowitz condition*, Trans. Amer. Math. Soc. **366** (2013), 4919–4937.
- [27] B. Pellacci, M. Squassina, *Mountain pass solutions for quasi-linear equations via a monotonicity trick*, J. Math. Anal. Appl. **381** (2011), 857–865.
- [28] J. Santos, S. Soares, *Radial solutions of quasilinear equations in Orlicz-Sobolev type spaces*, J. Math. Anal. Appl. **428** (2015), 1035–1053.
- [29] J. Serrin, H. Zou, *Symmetry of ground states of quasilinear elliptic equations*, Arch. Rat. Mech. Anal. **148** (1999), 265–290.
- [30] W. A. Strauss, *Existence of solitary waves in higher dimensions*, Comm. Math. Phys. **55** (1977), 149–162.
- [31] J. Su, Z. Q. Wang, M. Willem, *Weighted Sobolev embedding with unbounded and decaying radial potentials*, J. Diff. Eqns. **238** (2007), 201–219.
- [32] G. Talenti, *Best constant in Sobolev inequality*, Ann. Mat. Pura Appl. **110** (1976), 353–372.

- [33] N. S. Trudinger, *On Harnack type inequalities and their applications to quasilinear elliptic equations*, Comm. Pure Appl. Math. **20** (1967), 721–747.

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